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# EQUATIONS FOR MOMENTS AND STABILITY CONDITIONS OF LINEAR SYSTEMS WITH SCALAR PARAMETRIC PERTURBATION BY MARKOV CHAIN*

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Linear systems of differential equations with constant coefficients are considered. The structure of first and second moment equations is investigated under conditions of coefficient perturbation by a Markovian process with a finite number of states. A method is proposed for analyzing the stability of first and second moments by constructing transfer functions. The method is illustrated on the example of a second order equation perturbed by a symmetric telegraph signal.

1. Introduction. Foundations of the analysis of stability of differential equations with Markovian perturbation were established in /1/. A system of $n$ linear differential equations

$$\mathbf{x}^{\cdot} = \mathbf{A}(\xi)\mathbf{x} \tag{1.1}$$

acted upon by a uniform Markov chain $\xi_t$ with continuous time, a finite set of states $\{h_1, \ldots, h_N\}$, and an infinitesimal matrix $\mathbf{Q} = \{q_{ij}\}$ was considered in /2/. It was shown in /2,3/ that the analysis of the mean square stability of the system of stochastic differential equations (1.1) can be reduced to the analysis of stability of the determinate system of linear differential equations with constant coefficients

$$\mathbf{M}_r^{\cdot}(t) = \mathbf{A}(h_r)\mathbf{M}_r(t) + \mathbf{M}_r(t)\mathbf{A}^T(h_r) + \sum_{j=1}^{N} q_{jr}\mathbf{M}_j(t), \quad r = 1, \ldots N \tag{1.2}$$

where $\mathbf{M}_r(t)$ are symmetric matrices of order $n$. It was established in /2/ that the mean quadratic stability of system (1.1) is equivalent to the asymptotic stability of the trivial solution of the matrix system (1.2).

It is important to note that

$$\langle \mathbf{x}\mathbf{x}^T \rangle = \sum_{r=1}^{N} \mathbf{M}_r \tag{1.3}$$

which indicates the practical possibility of constructing a determinate system whose output would be represented by the matrix of second moments of the input stochastic system. Reasoning similar to that in /2,3/ makes it possible to obtain a determinate system of linear differential equations with constant coefficients for the determination of first moments

$$\mathbf{m}_r^{\cdot}(t) = \mathbf{A}(h_r)\mathbf{m}_r(t) + \sum_{j=1}^{N} q_{jr}\mathbf{m}_j(t), \quad r = 1, \ldots, N \tag{1.4}$$

with vectors $\mathbf{m}_r(t)$ such that

$$\langle \mathbf{x} \rangle = \sum_{r=1}^{N} \mathbf{m}_r \tag{1.5}$$

Stability of system (1.4) is the necessary condition for the mean quadratic stability of the original system (1.1).

The structure of systems (1.2) and (1.4) is investigated below in the most important in practice particular case of linear dependence of matrix $\mathbf{A}(\xi)$ on the perturbation $\xi$

$$\mathbf{A}(\xi) = \mathbf{A} + \xi\mathbf{b}\mathbf{c}^T \tag{1.6}$$

where $\mathbf{A}$ is a constant matrix of order $n \times n$, and $\mathbf{b}$ and $\mathbf{c}$ are constant vectors of order $n$. Constraint (1.6) enables us to derive simple stability conditions for system (1.4) in terms of the transfer function $\chi(p) = \mathbf{c}^T(p\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$ of the linear block

$$\mathbf{y}^{\cdot} = \mathbf{A}\mathbf{y} + \mathbf{b}v, \quad u = \mathbf{c}^T\mathbf{y} \tag{1.7}$$

from the input $v$ to the output $u$ in terms of the specified characteristics of the perturbation $\xi_t$. The proposed here method of stability investigation of system (1.4) of first moments can be successfully applied to the analysis of system (1.2) of second moments, and for obtaining the final stability conditions using the transfer matrix function $\mathbf{X}(p)$ of the linear block

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$$\mathbf{Y}^{\cdot} = \mathbf{AY} + \mathbf{YA}^T + \mathbf{bV}^T + \mathbf{Vb}^T, \ \mathbf{U} = \mathbf{Yc} \tag{1.8}$$

from the input vector $\mathbf{V}$ to the output vector $\mathbf{U}$. A byproduct of this investigation is the establishment of the existence of linear nondegenerate transformations of the phase coordinate systems (1.4) and (1.2) that split these systems in $N$ uniform blocks of the form (1.7) and (1.8). The use of these transforms simplifies the structure of systems (1.4) and (1.2), with the first moment vector $\langle\mathbf{x}\rangle$ and second moment matrix $\langle\mathbf{xx}^T\rangle$ appearing directly in the set of their new phase variables. In the most important case in which matrix $\mathbf{Q}$ has a simple structure, systems (1.4) and (1.2) may be presented, respectively, as

$$u_i = \chi(p - \lambda_i) v_i, \quad v_i = \sum_{j=1}^{N} \varphi_{ij} u_j, \quad i = 1, \ldots, N \tag{1.9}$$

$$\mathbf{U}_i = \mathbf{X}(p - \lambda_i) \mathbf{V}_i, \quad \mathbf{V}_i = \sum_{i=1}^{N} \varphi_{ij} \mathbf{U}_j, \quad i = 1, \ldots, N \tag{1.10}$$

where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of matrix $\mathbf{Q}$, and matrix $\mathbf{\Phi} = \{\varphi_{ij}\}$ is constructed in conformity with the specified Markov chain characteristics $\xi_t$. The stability of the linear determinate systems (1.9) and (1.10) define the mean and the quadratic mean stability of the investigated stochastic system.

The basic difference between the proposed here method of investigation of the stochastic systems (1.1) and (1.6) and the methods used in /2/, thus, consists in the analysis of the block structure of equations of moments. In the authors' opinion the analysis using transfer block functions also facilitates the second step of investigation, viz. the determination of spectral characteristics of the stationary random process in its transition through a linear block which is parametrically perturbed by a Markov chain. The latter problem requires separate investigation and is not considered here.

## 2. Investigation of the structure of equations for moments.

First, let us strictly define vectors $m_r(t)$ and matrices $M_r(t)$

$$\mathbf{m}_r(t) = \langle \mathbf{x}_t \delta(h_r - \xi_t) \rangle, \quad \delta(s) = \begin{cases} 1, & s = 0 \\ 0, & s \neq 0 \end{cases} \tag{2.1}$$

$$\mathbf{M}_r(t) = \langle \mathbf{x}_t \mathbf{x}_t^T \delta(h_r - \xi_t) \rangle, \quad r = 1, \ldots, N \tag{2.2}$$

where $\mathbf{x}_t$ is the solution of the stochastic system (1.1). The equalities (1.3) and (1.5) immediately follow from the definitions (2.1) and (2.2).

**Theorem 1.** Let vector $\mathbf{x}_0$ and numbers $P_1(0), \ldots, P_N(0)$ be the initial conditions of the stochastic system (1.1): $\mathbf{x}_0 = \mathbf{x}(0)$, $P_r(0) = P(\xi_0 = h_r)$. Then the set of vectors (2.1) $\mathbf{m}_1(t), \ldots, \mathbf{m}_N(t)$ is the solution of the system of Eqs. (1.4) with initial conditions $\mathbf{m}_r(0) = \mathbf{x}_0 p_r(0)$, $r = 1, \ldots, N$ and the set of matrices $\mathbf{M}_1(t), \ldots, \mathbf{M}_N(t)$ is the solution of the system of Eqs. (1.2) with initial conditions $\mathbf{M}_r(0) = \mathbf{x}_0 \mathbf{x}_0^T P_r(0)$, $r = 1, \ldots, N$. In conformity with assumption (1.6) the original system assumes the form

$$\mathbf{x}^{\cdot} = \mathbf{Ax} + \xi_t \mathbf{bc}^T \mathbf{x} \tag{2.3}$$

Let us investigate the structure of equations for the moments of solution of system (2.3). By virtue of (1.6) systems (1.4) and (1.2) become

$$\mathbf{m}_r^{\cdot}(t) = \mathbf{A}\mathbf{m}_r(t) + h_r \mathbf{bc}^T \mathbf{m}_r(t) + \sum_{j=1}^{N} q_{jr} \mathbf{m}_j(t) \tag{2.4}$$

$$\mathbf{M}_r^{\cdot}(t) = \mathbf{L}_a \mathbf{M}_r(t) + h_r \mathbf{L}_{bc} \mathbf{M}_r(t) + \sum_{j=1}^{N} q_{jr} \mathbf{M}_j(t), \quad \mathbf{L}_a \mathbf{Z} = \mathbf{AZ} + \mathbf{ZA}^T, \ \mathbf{L}_{bc} \mathbf{Z} = \mathbf{bc}^T \mathbf{Z} + \mathbf{Zcb}^T \tag{2.5}$$

where $\mathbf{L}_a$ and $\mathbf{L}_{bc}$ are linear operators acting in the space of symmetric matrices $\mathbf{Z}$ of order $n$.

**Theorem 2.** Let the infinitesimal matrix $\mathbf{Q}$ be reducible to the diagonal form. We denote by $\mathbf{d}_1, \ldots, \mathbf{d}_N$ the eigenvectors of matrix $\mathbf{Q}$ that correspond to eigenvalues $\lambda_1, \ldots, \lambda_N$ and compose matrix $\mathbf{D} = (\mathbf{d}_1, \ldots, \mathbf{d}_N) = \{d_{ij}\}_1^N$. Then, as the result of the linear transformation

$$\mathbf{y}_i = \sum_{r=1}^{N} \mathbf{m}_r d_{ri}, \quad i = 1, \ldots, N \tag{2.6}$$

system (2.4) for first moments splits in $N$ linear blocks

$$\mathbf{y}_i^{\cdot} = (\mathbf{A} + \lambda_i \mathbf{I}_n) \mathbf{y}_i + \mathbf{b} v_i, \ u_i = \mathbf{c}^T \mathbf{y}_i, \ i = 1, \ldots, N \tag{2.7}$$

linked by relations

$$v_i = \sum_{j=1}^{N} \varphi_{ij} u_j \tag{2.8}$$

and system (2.5) for second moments, after the similar transformation

$$\mathbf{Y}_i = \sum_{r=1}^{N} \mathbf{M}_r d_{ri}, \quad i = 1, \ldots, N \tag{2.9}$$

splits into the following blocks:

$$\mathbf{Y}_i^{\cdot} = (\mathbf{L}_a + \lambda_i \mathbf{I}_n)\,\mathbf{Y}_i + \mathbf{L}_b \mathbf{V}_i,\ \mathbf{U}_i^T = \mathbf{c}^T \mathbf{Y}_i,\ i = 1, \ldots, N,\ \ \mathbf{L}_b \mathbf{z} = \mathbf{b}\mathbf{z}^T + \mathbf{z}\mathbf{b}^T \tag{2.10}$$

with the connection relations

$$\mathbf{V}_i = \sum_{j=1}^{N} \varphi_{ij} \mathbf{U}_j, \quad i = 1, \ldots, N \tag{2.11}$$

where

$$\mathbf{\Phi} = \{\varphi_{ij}\} = \mathbf{D}^T \mathbf{H} (\mathbf{D}^T)^{-1}, \quad \mathbf{H} = \operatorname{diag}[h_1, \ldots, h_N] \tag{2.12}$$

and $L_b$ is a linear operator which maps the space of vectors of order $n$ into the space of symmetric matrices of order $n$.

The layout of connections between blocks of system (2.7), (2.8) is shown in Fig.1 for the case of perturbation by a Markov chain with three states. The following notation is used

$$\Sigma_i = \sum_{j=1}^{3} \varphi_{ij} u_j, \quad \chi_i = \chi(p - \lambda_i)$$

The layout of system (2.10), (2.11) is the same.

The layouts of system (2.7), (2.8) and (2.10), (2.11) enable us to make the following observations.

1°. Each of blocks (2.7) has a scalar input $v_i$ and output $u_i$, and is defined by a system of differential equations of order $n$.

2°. Each of blocks (2.10) has a vector input $\mathbf{V}_i$ and output $\mathbf{U}_i$ of order $n$, and is defined by a system of differential equations of order $n(n+1)/2$.

3°. The first blocks of systems (2.7), (2.8) and (2.10), (2.11) have as their phase coordinates, vector $\mathbf{y}_1 = \langle \mathbf{x} \rangle$ of first moments and matrix $\mathbf{Y}_1 = \langle \mathbf{x}\mathbf{x}^T \rangle$ of second moments, respectively, of initial stochastic system (2.3) (assuming that $\lambda_1 = 0$, $\mathbf{d}_1^T = (1, \ldots, 1)$).

4°. The linear transforms of phase variables (2.6) and (2.9) are nondegenerate, hence the stability of systems (2.4) and (2.5) is equivalent to the stability of systems (2.7), (2.8) and (2.10), (2.11) respectively.

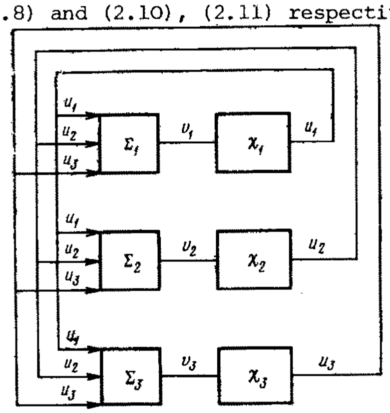

Fig.1

These observations indicate the possiblity of investigating the stability of the initial system (2.3) with respect to first and second moments by analyzing transfer functions and matrix functions of the linear blocks (2.7) and (2.10).

Similar block representations of systems (2.4) and (2.5) can be also obtained when matrix $\mathbf{Q}$ cannot be reduced to a diagonal form. For this it is necessary to take as the linear transformation matrix $\mathbf{D}$ as the matrix which reduces matrix $\mathbf{Q}$ to Jordan's form.

## 3. Stability with respect to first and second moments.

Let us define more precisely the considered here notion of stability.

**Definition 3.1.** We shall call system (1.1) (or 2.3) stable with respect to the first (second) moment, when the trivial solution of the linear system (1.4) (linear system (1.2)) is as a whole asymptotically stable.

Since systems (1.4) and (1.2) are linear and have constant coefficients, stability with respect to the first (second) moment means that $\langle x_t \rangle \to 0$ as $t \to \infty$ ($\langle x_t x_t^T \rangle \to 0$ as $t \to \infty$) and the order of approach to zero is exponential. According to /2/ stability with resepct to second moment is equivalent to asymptotic stability in the quadratic mean. It should be noted that the defined above stability with respect to the first moment is necessary for the asymptotic stability in the mean and all the more so for that in the mean square (see stability definitions in /4,5/).

**Theorem 3.** Let the infinitesimal matrix $\mathbf{Q}$ be of simple structure, $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of $\mathbf{Q}$, matrix $\mathbf{D} = (\mathbf{d}_1, \ldots, \mathbf{d}_N)$ be composed of respective eigenvectors of $Q$, matrices $\mathbf{H}$ and $\mathbf{\Phi}$ be defined by formulas (2.12), and $\chi(p)$ and $\mathbf{X}(p)$ be the transfer functions of linear blocks (1.7) and (1.8), respectively, $\Delta_1(p) = \det(p\mathbf{I} - \mathbf{A})$ and $\Delta_2(p)$ be the characteristic polynomials of the matrix differential equation in (1.8) of order $n(n+1)/2$. For stability of the stochastic system (2.3) with respect to the first moment it is, then, necessary and sufficient that the polynomial

$$\Delta_1(p - \lambda_1) \ldots \Delta_1(p - \lambda_N) \det [\mathbf{I}_N - \operatorname{diag}[\chi(p - \lambda_1), \ldots, \lambda(p - \lambda_N)] \mathbf{\Phi}] \tag{3.1}$$

be a Hurwitz polynomial, i.e. that all of its zeros lie to the left of the imaginary axis in the complex plane. For stability of the stochastic system (2,3) with respect to the second moment it is necessary and sufficient that the polynomial

$$\Delta_2(p - \lambda_1) \ldots \Delta_2(p - \lambda_N) \det [\mathbf{I}_{Nn} - \operatorname{diag}[\mathbf{X}(p - \lambda_1), \ldots, \mathbf{X}(p - \lambda_N)] \mathbf{\Phi} \otimes \mathbf{I}_n] \tag{3.2}$$

be a Hurwitz polynomial. In this formula $\mathbf{\Phi} \otimes \mathbf{I}_n$ denotes the direct product of matrices $\mathbf{\Phi}$ and $\mathbf{I}_n$ (see /6/).

The conditions of Theorems 2 and 3 disregard the case when matrix $\mathbf{Q}$ cannot be reduced to a diagonal form. However the application of the linear transform which reduces $\mathbf{Q}$ to Jordan's form yields results similar to those obtained above.

**4. Determination of the matrix transfer function.** The stipulated in Theorem 3 conditions of stability with respect to the second moment presumed that the transfer matrix function $\mathbf{X}(p)$ of the linear block (1.8) is known between the input $\mathbf{V}$ and output $\mathbf{U}$. This function is a square matrix of order $n$ whose elements are proper rational fraction whose denominator is of power $n(n+1)/2$. We shall indicate two methods for its derivation.

The first method uses the vector representation of block (1.8). Such representation can always be obtained by forming the vector of independent phase coordinates $\mathbf{z}$ from $n(n+1)/2$ elements of matrix $\mathbf{Y}$ lying on the principal diagonal and above it, i.e.

$$z^{\cdot} = \mathbf{A}_1\mathbf{z} + \mathbf{B}\mathbf{V}, \quad \mathbf{U} = \mathbf{C}^T\mathbf{z} \tag{4.1}$$

For instance, when $n = 2$ we have in this representation

$$\mathbf{z} = \begin{Vmatrix} y_{11} \\ y_{12} \\ y_{22} \end{Vmatrix}, \quad \mathbf{A}_1 = \begin{Vmatrix} 2a_{11} & 2a_{12} & 0 \\ a_{21} & a_{11} + a_{22} & a_{12} \\ 0 & 2a_{21} & 2a_{22} \end{Vmatrix}, \quad \mathbf{B} = \begin{Vmatrix} 2b_1 & 0 \\ b_2 & b_1 \\ 0 & 2b_2 \end{Vmatrix}, \quad \mathbf{C} = \begin{Vmatrix} c_1 & 0 \\ c_2 & c_1 \\ 0 & c_2 \end{Vmatrix}$$

where $a_{ij}$, $b_i$, and $c_j$ are, respectively, elements of matrix $\mathbf{A}$ and of vectors $\mathbf{b}$ and $\mathbf{c}$.

Using the representation (4.1) we obtain $\mathrm{X}(p) = C^T(p\mathbf{I}_{n(n+1)/2} - \mathrm{A}_1)^{-1}\mathbf{B}$. Unfortunately, as the order $n$ is increased, the dimensions of matrices $\mathbf{A}_1$, $\mathbf{B}$, and $\mathbf{C}$ rapidly increase, and the formulas defining them increase in complexity. Hence, means for determining the transfer matrix $\mathbf{X}(p)$ without resorting to the vector representation (4.1) is of interest.

The second method consists of solving the Liapunov matrix equation, as proposed in /7/. The transfer matrix defines the dependence between Laplace representation of the input and output of block (1.8) as follows: $\mathbf{U} = \mathbf{X}(p)\mathbf{V}$. The same dependence is defined by the relations

$$p\mathbf{Y} = \mathbf{A}\mathbf{Y} + \mathbf{Y}\mathbf{A}^T + \mathbf{b}\mathbf{V}^T + \mathbf{V}\mathbf{b}^T, \quad \mathbf{U} = \mathbf{Y}\mathbf{c}$$

Hence columns $\mathbf{X}_i(p)$ and matrix $[\mathbf{X}_1(p), \ldots, \mathbf{X}_n(p)] = X(p)$ can be determined by solving the matrix algebraic equations

$$\left(\mathbf{A} - \frac{p}{2}\mathbf{I}\right)\mathbf{Y} + \mathbf{Y}\left(\mathbf{A} - \frac{p}{2}\mathbf{I}\right)^T = -(\mathbf{b}\mathbf{e}_i^T + \mathbf{e}_i\mathbf{b}^T), \quad \mathbf{X}_i(p) = \mathbf{Y}\mathbf{c}, \quad i = 1, \ldots, n \tag{4.2}$$

where $\mathbf{e}_i$ is a vector of order $n$ whose coordinate number $i$ is unity and the remaining ones are zero. According to /7/ the following operations must be carried out for calculating columns $\mathbf{X}_i(p)$.

1. Determine the coefficients $f_j(p)$ and $\mathbf{F}_j(p)$ of the scalar and matrix polynomials $f(\lambda, p)$ and $\mathbf{F}(\lambda, p)$

$$f(\lambda, p) = \lambda^n + f_1(p)\lambda^{n-1} + \ldots + f_n(p) = \det\left(\left(\lambda + \frac{p}{2}\right)\mathbf{I} - \mathbf{A}\right)$$

$$\mathbf{F}(\lambda, p) = \mathbf{F}_1(p)\lambda^{n-1} + \ldots + \mathbf{F}_n(p) = f(\lambda, p)\left(\left(\lambda + \frac{p}{2}\right)\mathbf{I} - \mathbf{A}\right)^{-1}$$

Coefficients of the matrix polynomial $F(\lambda, p)$ can be calculated using the recurrent formulas

$$\mathbf{F}_1(p) = \mathbf{I}, \quad \mathbf{F}_2(p) = \left(\mathbf{A} - \frac{p}{2}\mathbf{I}\right)\mathbf{F}_1(p) + f_1(p)\mathbf{I}, \ \ldots, \ \mathbf{F}_n(p) = \left(\mathbf{A} - \frac{p}{2}\mathbf{I}\right)\mathbf{F}_{n-1}(p) + f_{n-1}\mathbf{I}$$

2. Determine the first row $[\alpha_1(p), \dots, \alpha_n(p)]$ of matrix $[\mathbf{H}_n(f)]^{-1}$, where $\mathbf{H}_n(f)$ is the Hurwitz matrix of the polynomial $f(\lambda, p)$

$$\mathbf{H}_n(f) = \begin{Vmatrix} f_1(p) & 1 & 0 & \dots & 0 \\ f_3(p) & f_2(p) & f_1(p) & \dots & 0 \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \dots & \cdot\ \cdot \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \dots & \cdot\ \cdot \\ 0 & 0 & 0 & \dots & f_{n-2}(p) \\ 0 & 0 & 0 & \dots & f_n(p) \end{Vmatrix}$$

3. Determine the $n$-dimensional columns

$$\begin{Vmatrix} \mathbf{g}_1(p) \\ \mathbf{g}_2(p) \\ \cdot \\ \cdot \\ \mathbf{g}_n(p) \end{Vmatrix} = \begin{Vmatrix} \mathbf{F}_1(p) & 0 & 0 & \dots & 0 \\ \mathbf{F}_3(p) & \mathbf{F}_2(p) & \mathbf{F}_1(p) & \dots & 0 \\ \cdot\ \cdot & \cdot\ \cdot & \cdot & \dots & \cdot\ \cdot \\ \cdot\ \cdot & \cdot\ \cdot & \cdot & \dots & \cdot\ \cdot \\ 0 & 0 & 0 & & \mathbf{F}_n(p) \end{Vmatrix} \times \begin{Vmatrix} (\mathbf{b}\mathbf{e}_i^T + \mathbf{e}_i\mathbf{b}^T)\,\mathbf{F}_1^T(p)\,\mathbf{c} \\ -(\mathbf{b}\mathbf{e}_i^T + \mathbf{e}_i\mathbf{b}^T)\,\mathbf{F}_2^T(p)\,\mathbf{c} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ (-1)^{n-1}(\mathbf{b}\mathbf{e}_i^T + \mathbf{e}_i\mathbf{b}^T)\,\mathbf{F}_n^T(p)\,\mathbf{c} \end{Vmatrix}$$

4. Calculate the column

$$\mathbf{X}_i(p) = \frac{1}{2}\sum_{j=1}^{n} \alpha_j(p)\,\mathbf{g}_j(p)$$

**5. Example.** Let us consider the differential equation

$$x^{\cdot\cdot} + (a_1 + \alpha\xi)x^{\cdot} + (a_2 + \beta\xi)x = 0 \tag{5.1}$$

perturbed by a symmetric telegraph signal, which is assumed to be a Markov chain $\xi_t$ with two states $h$ and $-h$ and the infinitesimal matrix

$$\mathbf{Q} = \begin{Vmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{Vmatrix}$$

where the numbers $h$ and $\lambda$ are positive. In this case the structure of matrix $\mathbf{Q}$ is simple and its eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -2\lambda$, to which correspond eigenvectors $\mathbf{d}_1 = (1, 1)^T$ and $\mathbf{d}_2 = (1, -1)^T$. In conformity with definition (2.12) we obtain the matrix

$$\mathbf{\Phi} = h\begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$$

To determine stability with respect to the first moment we determine the transfer function of system (5.1) from the input $v = \alpha\xi x^{\cdot} + \beta\xi x$ to the output $u = \alpha x + \beta x$

$$\chi(p) = \frac{\gamma(p)}{\Delta_1(y)} = -\frac{\alpha p + \beta}{p^2 + a_1 p + a_2}$$

The polynomial (3.1) assumes the form

$$\Delta_1(p)\Delta_1(p + 2\lambda) - h^2\gamma(p)\gamma(p + 2\lambda) \tag{5.2}$$

Equation (5.1) is stable with respect to the first moment, when (5.2) is a Hurwitz polynomial.

It is interesting to note that the asymptotic stability of the trivial solution of the unperturbed equation $x^{\cdot\cdot} + a_1x^{\cdot} + a_2x = 0$ is not a necessary condition of stability of (5.1) with respect to the first moment. Thus, for example, the equation

$$x^{\cdot\cdot} + (1 + \xi)x^{\cdot} - \lambda\xi x = 0 \tag{5.3}$$

is stable with respect to the first moment for any $\lambda > 0$ and $h = 1$, which means that $\langle x_t\rangle \to 0$ and $\langle x_t^{\cdot}\rangle \to 0$ as $t \to \infty$. Nevertheless the trivial solution of the unperturbed equation $x^{\cdot\cdot} + x^{\cdot} = 0$ is not asymptotically stable. Moreover the realization of solution of the perturbed equation (5.3) depends in some time intervals on the unstable equation $x^{\cdot\cdot} + 2x^{\cdot} - \lambda x = 0$, and in others on the asymptotically unstable equation $x^{\cdot\cdot} + \lambda x = 0$.

To investigate Eq. (5.1) with respect to the second moment we determine the transfer matrix $\mathbf{X}(p)$ of the linear block (1.8) in which

$$\mathbf{A} = \begin{Vmatrix} 0 & 1 \\ -a_2 & -a_1 \end{Vmatrix}, \quad \mathbf{b} = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}, \quad \mathbf{c} = \begin{Vmatrix} -\beta \\ -\alpha \end{Vmatrix}$$

In accordance with Sect.4 we have

$$\mathbf{X}(p)=\frac{\varkappa(p)}{\Delta_2(p)},\quad \varkappa(p)=\left\|\begin{matrix}(p+2a_1)(\alpha p+2\beta) & 2\alpha p+4\beta\\ p(-2\alpha a_2+\beta(p+2a_1)) & 2\alpha(p^2+a_1p+2a_2)+2\beta p\end{matrix}\right\|$$

$$\Delta_2(p)=(p+a_1)(p^2+2a_1p+4a_2)$$

and polynomial (3.2) assumes the form

$$\Delta_2(p)\Delta_2(p+2\lambda)-h^2\,\mathrm{sp}\,[\varkappa(p+2\lambda)\varkappa(p)]+h^4 4\alpha^2(\alpha p+2\beta)(\alpha(p+2\lambda)+2\beta) \tag{5.4}$$

Equation (5.1) is stable with respect to the second moment, when (5.4) is a Hurwitz polynomial.

For Eq. (5.3) polynomial (5.4) cannot be a Hurwitz polynomial, since its free term is $(-16\lambda^3(\lambda+1))$ and, consequently, negative for all $\lambda>0$. Thus Eq. (5.3) cannot be stable with respect to the second moment.

## 6. Proof of Theorems 1—3.

For Theorem 1 we shall prove only the first statement, since the second was proved in /2,3/. We write system (1.1) in the integral form

$$\mathbf{x}_t=\mathbf{x}_0+\int_0^t \mathbf{A}(\xi_\tau)\,\mathbf{x}_\tau\,d\tau$$

multiply both of its parts by function $\delta(h_r-\xi_t)$ (see (2.1)), and apply the operation of mathematical expectation

$$\mathbf{m}_r(t)=\mathbf{x}_0p_r(t)+\int_0^t<\mathbf{A}(\xi_\tau)\,\mathbf{x}_\tau\delta(h_r-\xi_t)>d\tau$$

To obtain a system of integral equations for functions $\mathbf{m}_r(t)$ we transform the integrand of this equation.

We apply the operation of conditional mathematical expectation $(\langle\cdot\rangle=\langle\cdot\mid\xi_\tau\rangle)$ for $\tau\leqslant t$

$$\begin{aligned}\langle\mathbf{A}(\xi_\tau)\,\mathbf{x}_\tau\delta(h_r-\xi_t)\rangle&=\sum_{j=1}^N\langle\mathbf{A}(\xi_\tau)\,\mathbf{x}_\tau\delta(h_j-\xi_\tau)\,\delta(h_r-\xi_t)\rangle=\\ &\sum_{j=1}^N\langle\mathbf{A}(\xi_\tau)\,\mathbf{x}_\tau\delta(h_j-\xi_\tau)\langle\delta(h_r-\xi_t)\mid\xi_\tau\rangle\rangle=\\ &\sum_{j=1}^N\langle\mathbf{A}(h_j)\,\mathbf{x}_\tau\delta(h_j-\xi_\tau)\langle\delta(h_r-\xi_t)\mid\xi_\tau=h_j\rangle\rangle=\\ &\sum_{j=1}^N\mathbf{A}(h_j)\langle\mathbf{x}_\tau\delta(h_j-\xi_\tau)\rangle\langle\delta(h_r-\xi_t)\mid\xi_\tau=h_j\rangle=\\ &\sum_{j=1}^N\mathbf{A}(h_j)\,\mathbf{m}_j(\tau)\,\pi_{jr}(t-\tau)\end{aligned}$$

where $\{\pi_{jr}(t)\}_{j,r=1}^N=\pi(t)$ is the matrix of transient probabilities of the Markov chain during time $t$.

It follows from this that the vector functions $m_r(t)$ satisfy the system

$$\mathbf{m}_r(t)=\mathbf{x}_0p_r(t)+\int_0^t\sum_{j=1}^N\mathbf{A}(h_j)\,\mathbf{m}_j(\tau)\,\pi_{jr}(t-\tau)\,d\tau,\quad r=1,\ldots,N \tag{6.1}$$

Differentiation of the right- and left-hand sides of (6.1) yields the system of Eqs. (1.4).

To prove Theorems 2 and 3 we use the concept of direct product of matrices. All relevant properties of direct product of square matrices can be found in /6,8/. Using the notation of (2.12), $\mathbf{m}^T=\mathbf{m}^T(t)=[\mathbf{m}_1{}^T(t),\ldots,\mathbf{m}_N{}^T(t)]$, $\mathbf{M}^T=\mathbf{M}^T(t)=[\mathbf{M}_1(t),\ldots,\mathbf{M}_N(t)]$ and the symbol $\otimes$ for the direct product, we transform system to the form

$$\mathbf{m}^\cdot=(\mathbf{I}_N\otimes\mathbf{A})\,\mathbf{m}+(\mathbf{H}\otimes\mathbf{bc}^T)\,\mathbf{m}+(\mathbf{Q}^T\otimes\mathbf{I}_n)\,\mathbf{m} \tag{6.2}$$

$$\mathbf{M}^\cdot=(\mathbf{I}_N\otimes\mathbf{L}_a)\,\mathbf{M}+(\mathbf{H}\otimes\mathbf{L}_{bc})\,\mathbf{M}+(\mathbf{Q}^T\otimes\mathbf{I}_n)\,\mathbf{M} \tag{6.3}$$

Let us prove the first statement of Theorem 2, by obtaining system (2.7), (2.8) equivalent to system (2.4). We carry out the linear substitution $\mathbf{y}=(\mathbf{D}^T\otimes\mathbf{I}_n)\,\mathbf{m}$ in (2.6) and for the system of linear differential equations in new phase variables $\mathbf{y}^T=[\mathbf{y}_1{}^T,\ldots,\mathbf{y}_N{}^T]$ obtain

$$\mathbf{y}^\cdot=(\mathbf{D}^T\otimes\mathbf{I}_n)\,[(\mathbf{I}_N\otimes\mathbf{A})+(\mathbf{H}\otimes\mathbf{bc}^T)+(\mathbf{Q}\otimes\mathbf{I}_n)]\,[(\mathbf{D}^T)^{-1}\otimes\mathbf{I}_n]\,\mathbf{y} \tag{6.4}$$

Using the properties of the direct product and the notation

$$\boldsymbol{\Phi}=\mathbf{D}^T\mathbf{H}(\mathbf{D}^T)^{-1},\quad \boldsymbol{\Lambda}=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)$$

we obtain

$$y^{\cdot} = (I_N \otimes A)\,y + (\Phi \otimes b)\,(I_N \otimes c^T)\,y + (\Lambda \otimes I_n)\,y \tag{6.5}$$

By splitting system (6.5) into blocks we pass to the required relations (2.7) and (2.8). The second statement of Theorem 2 is similarly proved using vector representation of matrix equations.

To prove Theorem 3 it is sufficient to show that the chracteristic polynomial of linear systems (2.10), (2.11) and (2.7), (2.8) are, respectively, of form (3.2) and (3.1). According to Sect.4 the matrix system (2.10), (2.11) is linear, of order $n(n+1)N/2$, and its block vector representation is of the form

$$z_i^{\cdot} = (A_1 + \lambda_i I)\,z_i + BV_i,\quad U_i = C^T z_i,\quad i = 1, \ldots, N \tag{6.6}$$

with the connection relations (2.12) between blocks. Here $A_1$ is a square matrix of order $n(n+1)/2$, and $B$ and $C$ are rectangular matrices of dimensions $n(n+1)/2 \times n$, respectively. Systems (2.7), (2.8) and (6.6), (2.11) are of the same type, differing only in the dimensions of the input and output phase vectors. We restrict the calculation to the determination of the characteristic polynomial of system (2.7), (2.8). Using the notation

$$I_N \otimes A + \Lambda \otimes I_n = A_2,\quad \Phi \otimes b = B_2,\quad I_N \otimes c^T = C_2^T$$

we represent system (6.5) in the form

$$y^{\cdot} = (A_2 + B_2 C_2^T)\,y \tag{6.7}$$

Let us determine the characteristic polynomial. It is shown in /8/ that

$$\det[pI_{nN} - (A_2 + B_2 C_2^T)] = \det(pI_{nN} - A_2)\cdot \det[I_N - C_2^T (pI_{nN} - A_2)^{-1} B_2]$$

Then

$$\det(pI_{nN} - A_2) = \Delta_1(p - \lambda_1) \ldots \Delta_1(p - \lambda_N),\quad C_2^T (pI_{nN} - A_2)^{-1} B_2 = \operatorname{diag}[\chi(p - \lambda_1), \ldots, \chi(p - \lambda_N)]\,\Phi$$

Thus $\det[pI_{nN} - (A_2 + B_2 C_2^T)]$ is equal to polynomial (3.1), Q.E.D.

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